Kuwait University | **Calculus II : Math 102** | 11 December 2011
Department of Mathematics | **Second Examination** | Duration : 90 minutes

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The use of a calculator of any kind is not allowed. All communication devices including mobile telephones should be switched off. Answer all of the following questions.

Evaluate the following. **($2^1/_2$ points each)**

1. $\displaystyle\int \arctan(1/x)\,dx$

2. $\displaystyle\int \frac{\sin(2x)}{\sec^3 x}\,dx$

3. $\displaystyle\int \frac{1}{(\cos x)-1}\,dx$

4. $\displaystyle\int_0^{\pi/8} \frac{1-\tan^2 x}{\sec^2 x}\,dx$

5. $\displaystyle\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$

6. $\displaystyle\int \frac{1}{\sqrt{e^x-1}}\,dx$

7. $\displaystyle\int \frac{x^2+x+1}{x^2-1}\,dx$

8. $\displaystyle\int \frac{1}{x^2+2x+2}\,dx$

9. $\displaystyle\int_0^{9} \frac{1}{\sqrt[3]{x-1}}\,dx$

10. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{4+x^2}\,dx$

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SOLUTION

1. Integrate by parts, with $u = \arctan(1/x)$ and $dv = dx$.

$$\frac{du}{dx} = \frac{1}{1+(1/x)^2}\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{1}{1+(1/x)^2}\left(-\frac{1}{x^2}\right) = -\frac{1}{x^2+1}.$$

So $du = -1/(x^2+1)\,dx$ and $v = x$. Hence,

$$\int \arctan(1/x)\,dx = x\arctan(1/x) + \int \frac{x}{x^2+1}\,dx = x\arctan(1/x) + \tfrac{1}{2}\ln(x^2+1) + C.$$

2. Simplify,

$$\int \frac{\sin 2x}{\sec^3 x}\,dx = 2\int \sin x \cos^4 x\,dx.$$

Substitute $u = \cos x$. So $du = -\sin x\,dx$. Hence,

$$\int \frac{\sin 2x}{\sec^3 x}\,dx = -2\int u^4\,du = -\tfrac{2}{5}u^5 + C = -\tfrac{2}{5}\cos^5 x + C.$$

3. Method 1. Use the Method of Weierstrass. Substitute $t = \tan(x/2)$, so that $\cos x = (1-t^2)/(1+t^2)$ and $dx = 2/(1+t^2)\,dt$. Then

$$\begin{aligned}\int \frac{1}{\cos x - 1}\,dx &= \int \frac{1}{(1-t^2)/(1+t^2) - 1}\,\frac{2}{1+t^2}\,dt = \cdots = -\int \frac{1}{t^2}\,dt \\ &= \frac{1}{t} + C = \cot(x/2) + C.\end{aligned}$$

Method 2.

$$\int \frac{1}{\cos x - 1}\,dx = \int \frac{1}{-2\sin^2(x/2)}\,dx = -\tfrac{1}{2}\int \csc^2(x/2)\,dx = \cot(x/2) + C.$$

4.

$$\begin{aligned}\int_0^{\pi/8} \frac{1-\tan^2 x}{\sec^2 x}\,dx &= \int_0^{\pi/8} \cos^2 x - \sin^2 x\,dx = \int_0^{\pi/8} \cos 2x\,dx = \tfrac{1}{2}\sin 2x\Big|_0^{\pi/8} \\ &= \tfrac{1}{2}\sin(\pi/4) = \frac{1}{2\sqrt{2}}.\end{aligned}$$

5. Substitute $x = 5\sin\theta$. So, $\sqrt{25-x^2} = 5\cos\theta$ and $dx = 5\cos\theta\,d\theta$. Hence,

$$\begin{aligned}\int \frac{1}{x^2\sqrt{25-x^2}}\,dx &= \int \frac{1}{(5\sin\theta)^2}\,d\theta = \frac{1}{25}\int \csc^2\theta\,d\theta = -\frac{1}{25}\cot\theta + C \\ &= -\frac{5\cos\theta}{25(5\sin\theta)} + C = -\frac{\sqrt{25-x^2}}{25x} + C.\end{aligned}$$

6. Substitute $x = -2\ln u$. So, $dx = (-2/u)\,du$. Hence,

$$\begin{aligned}\int \frac{1}{\sqrt{e^x - 1}}\,dx &= \int \frac{1}{\sqrt{u^{-2}-1}}\,\frac{-2}{u}\,du = -2\int \frac{1}{\sqrt{1-u^2}}\,du = 2\arccos u + C \\ &= 2\arccos(e^{-x/2}) + C.\end{aligned}$$

7.
$$\frac{x^2+x+1}{x^2-1} = 1 + \frac{x+2}{x^2-1} = 1 + \frac{A}{x-1} + \frac{B}{x+1},$$

where

$$A(x+1) + B(x-1) = x + 2.$$

Substitution of $x = 1$ gives $A = 3/2$. Substitution of $x = -1$ gives $B = -1/2$. Hence,

$$\begin{aligned}\int \frac{x^2+x+1}{x^2-1}\,dx &= \int \left[1 + \frac{3}{2(x-1)} - \frac{1}{2(x+1)}\right] dx \\ &= x + \tfrac{3}{2}\ln|x-1| - \tfrac{1}{2}\ln|x+1| + C.\end{aligned}$$

8. Complete the square,

$$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{1+(x+1)^2}\,dx = \arctan(x+1) + C.$$

9. The integrand is discontinuous at $x = 1$.

$$\int_0^t \frac{1}{\sqrt[3]{x-1}}\,dx = \tfrac{3}{2}(x-1)^{2/3}\Big|_0^t = \tfrac{3}{2}\left[(t-1)^{2/3} - (-1)^{2/3}\right] \to -\tfrac{3}{2} \quad \text{as } t \to 1^-.$$

$$\int_t^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \tfrac{3}{2}(x-1)^{2/3}\Big|_t^9 = \tfrac{3}{2}\left[8^{2/3} - (t-1)^{2/3}\right] \to 6 \quad \text{as } t \to 1^+.$$

Hence,

$$\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = -\tfrac{3}{2} + 6 = \tfrac{9}{2}.$$

10.
$$\int_0^t \frac{1}{x^2+4}\,dx = \tfrac{1}{2}\arctan(x/2)\Big|_0^t = \tfrac{1}{2}\arctan(t/2) \to \frac{\pi}{4} \quad \text{as } t \to \infty.$$

Hence, since the integrand is even,

$$\int_{-\infty}^{\infty} \frac{1}{4+x^2}\,dx = 2\int_0^{\infty} \frac{1}{4+x^2}\,dx = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}.$$